Correction de l'examen final de Mécanique des Milieux Déformables du 12 janvier 2016

Questions de cours :

1) Dans le cas général, on a la formule :

$$\overset{=>}{\varepsilon} = \frac{1+\vartheta}{E}\overset{=>}{\sigma} - \frac{\vartheta}{E}\left(trace\ \overset{=>}{\sigma}\right)\overset{=>}{I}$$

Soient les 3 équations :

$$\begin{cases} \varepsilon_{xx} = \dfrac{\sigma_{xx}}{E} - \dfrac{\vartheta}{E}(\sigma_{yy} + \sigma_{zz}) \\ \varepsilon_{yy} = \dfrac{\sigma_{yy}}{E} - \dfrac{\vartheta}{E}(\sigma_{xx} + \sigma_{zz}) \\ \varepsilon_{xy} = \dfrac{1+\vartheta}{E}\sigma_{xy} \end{cases}$$

Pour notre cas, on a : $\sigma_{zz} = 0$; les équations précédentes se réduisent donc à :

$$\begin{cases} \varepsilon_{xx} = \dfrac{\sigma_{xx}}{E} - \dfrac{\vartheta}{E}(\sigma_{yy}) & (a) \\ \varepsilon_{yy} = \dfrac{\sigma_{yy}}{E} - \dfrac{\vartheta}{E}(\sigma_{xx}) & (b) \\ \varepsilon_{xy} = \dfrac{1+\vartheta}{E}\sigma_{xy} & (c) \end{cases}$$

$(a) \Rightarrow \ \sigma_{xx} = E\varepsilon_{xx} + \vartheta\sigma_{yy}$

$(b) \Rightarrow \ \sigma_{yy} = E\varepsilon_{yy} + \vartheta\sigma_{xx}$

$$(b)\ dans\ (a) \Rightarrow \ \sigma_{xx} = E\varepsilon_{xx} + \vartheta E\varepsilon_{yy} + \vartheta^2\sigma_{xx}$$
$$\Rightarrow (1-\vartheta^2)\sigma_{xx} = E\varepsilon_{xx} + E\vartheta\varepsilon_{yy}$$
$$\Rightarrow \boxed{\sigma_{xx} = \frac{E}{1-\vartheta^2}\ \varepsilon_{xx} + \frac{E\vartheta}{1-\vartheta^2}\varepsilon_{yy}}$$

$$(a)\ dans\ (b) \Rightarrow \ \sigma_{yy} = E\varepsilon_{yy} + \vartheta E\varepsilon_{xx} + \vartheta^2\sigma_{yy}$$
$$\Rightarrow (1-\vartheta^2)\sigma_{yy} = E\varepsilon_{yy} + E\vartheta\varepsilon_{xx}$$
$$\Rightarrow \boxed{\sigma_{yy} = \frac{E}{1-\vartheta^2}\ \varepsilon_{yy} + \frac{E\vartheta}{1-\vartheta^2}\varepsilon_{xx}}$$

$$(c) \Rightarrow \ \sigma_{xy} = \frac{E\varepsilon_{xy}}{1+\vartheta}$$
$$\Rightarrow \boxed{\sigma_{xy} = \frac{E}{2(1+\vartheta)}\vartheta_{xy}} \qquad car\ \varepsilon_{xy} = \frac{\vartheta_{xy}}{2}$$

2) $\varepsilon_{zz} = \frac{\sigma_{zz}}{E} - \frac{\vartheta}{E}(\sigma_{yy} + \sigma_{xx})$

Or ici, $\sigma_{zz} = 0$ donc :

$$\boxed{\varepsilon_{zz} = -\frac{\vartheta}{E}(\sigma_{yy} + \sigma_{xx})}$$

D'après (1), on remplace σ_{xx} et σ_{yy} par leurs valeurs en fonction de ε_{xx} et ε_{yy}

$$\varepsilon_{zz} = -\frac{\vartheta}{E}\left(\frac{E}{1-\vartheta^2}\,\varepsilon_{xx} + \frac{E\vartheta}{1-\vartheta^2}\varepsilon_{yy} + \frac{E}{1-\vartheta^2}\,\varepsilon_{yy} + \frac{E\vartheta}{1-\vartheta^2}\varepsilon_{xx}\right)$$

$$\varepsilon_{zz} = \frac{-\vartheta(1+\vartheta)}{1-\vartheta^2}\,(\varepsilon_{xx} + \varepsilon_{yy})$$

$$\boxed{\varepsilon_{zz} = \frac{-\vartheta}{1-\vartheta}\,(\varepsilon_{xx} + \varepsilon_{yy})}$$

3) Etat plan de déformation :

$$\overset{=>}{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{xy} & \varepsilon_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad et \quad \overset{=>}{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}$$

4) $\varepsilon_{ij}^{th} = \alpha\Delta T\delta_i = \alpha(T - T_0)\delta_{ij}$ avec T = T(x,y,z)

Ecrivons alors les 6 équations de Saint Venant :

$$\begin{cases} \frac{d^2\mathcal{E}_{xx}}{dy^2} + \frac{d^2\mathcal{E}_{yy}}{dx^2} - \frac{2d^2\mathcal{E}_{xy}}{dxdy} = 0 \\ \frac{d^2\mathcal{E}_{yy}}{dz^2} + \frac{d^2\mathcal{E}_{zz}}{dy^2} - \frac{2d^2\mathcal{E}_{yz}}{dydz} = 0 \\ \frac{d^2\mathcal{E}_{zz}}{dx^2} + \frac{d^2\mathcal{E}_{xx}}{dz^2} - \frac{2d^2\mathcal{E}_{xz}}{dxdz} = 0 \\ -\frac{d^2\mathcal{E}_{xx}}{dydz} + \frac{d}{dx}\left(\frac{d\mathcal{E}_{xy}}{dz} - \frac{d\mathcal{E}_{yz}}{dx} + \frac{d\mathcal{E}_{xz}}{dy}\right) = 0 \\ -\frac{d^2\mathcal{E}_{yy}}{dxdz} + \frac{d}{dy}\left(\frac{d\mathcal{E}_{yz}}{dx} - \frac{d\mathcal{E}_{xz}}{dy} + \frac{d\mathcal{E}_{xy}}{dz}\right) = 0 \\ -\frac{d^2\mathcal{E}_{zz}}{dxdy} + \frac{d}{dz}\left(\frac{d\mathcal{E}_{xz}}{dy} - \frac{d\mathcal{E}_{xy}}{dz} + \frac{d\mathcal{E}_{yz}}{dx}\right) = 0 \end{cases} \rightarrow \begin{cases} \alpha\frac{\partial^2 T}{\partial y^2} + \alpha\frac{\partial^2 T}{\partial x^2} - 0 = 0 \Rightarrow \frac{\partial^2 T}{\partial y^2} = -\frac{\partial^2 T}{\partial x^2} & (1) \\ \alpha\frac{\partial^2 T}{\partial z^2} + \alpha\frac{\partial^2 T}{\partial y^2} - 0 = 0 \Rightarrow \frac{\partial^2 T}{\partial y^2} = -\frac{\partial^2 T}{\partial z^2} & (2) \\ \alpha\frac{\partial^2 T}{\partial x^2} + \alpha\frac{\partial^2 T}{\partial z^2} - 0 = 0 \Rightarrow \frac{\partial^2 T}{\partial x^2} = -\frac{\partial^2 T}{\partial z^2} & (3) \\ -\frac{\partial^2 T}{\partial x\partial z} + 0 = 0 \Rightarrow \frac{\partial^2 T}{\partial x\partial z} = 0 & (4) \\ -\frac{\partial^2 T}{\partial y\partial z} + 0 = 0 \Rightarrow \frac{\partial^2 T}{\partial y\partial z} = 0 & (5) \\ -\frac{\partial^2 T}{\partial x\partial y} + 0 = 0 \Rightarrow \frac{\partial^2 T}{\partial x\partial y} = 0 & (6) \end{cases}$$

En utilisant 2 équations parmi (1),(2) et (3) on obtient :

$$-\frac{\partial^2 T}{\partial y^2} = \frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T}{\partial z^2} \qquad (*)$$

Et les égalités (4),(5) et (6) nous donnent :

$$\frac{\partial^2 T}{\partial x \partial z} = \frac{\partial^2 T}{\partial y \partial z} = \frac{\partial^2 T}{\partial x \partial y} = 0 \qquad (**)$$

On peut donc conclure d'après (*) et (**) la forme de T(x,y,z) :

$$T(x,y,z) = Ax^2 - Ay^2 + Az^2 + Bx + Cy + Dz + cte$$

Avec cte = T(0,0,0) = T_0 $et\ A, B, C, D \in R$

D'où finalement :

$$\boxed{T(x,y,z) = Ax^2 - Ay^2 + Az^2 + Bx + Cy + Dz + T_0}$$

<u>Problème 1 :</u>

$$\overset{=>}{\varepsilon}_{x,y,z} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{xy} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix}$$

1) On a la formule générale : $\varepsilon_k = \overrightarrow{n_k}.\overset{=>}{\varepsilon}.\overrightarrow{n_k}$
   $\overrightarrow{n_k}$ étant le vecteur unitaire dirigé suivant k

   Dans notre cas on a :

$$\varepsilon_a = \overrightarrow{n_a}.\overset{=>}{\varepsilon}.\overrightarrow{n_a} = < 1\,, 0 > \overset{=>}{\varepsilon} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \boxed{\Rightarrow\ \varepsilon_{xx} = 0.001}$$

$$\varepsilon_c = \overrightarrow{n_c}.\overset{=>}{\varepsilon}.\overrightarrow{n_c} = < 0\,, 1 > \overset{=>}{\varepsilon} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \boxed{\Rightarrow\ \varepsilon_{yy} = 0.0008}$$

$$\varepsilon_b = \overrightarrow{n_b}.\overset{=>}{\varepsilon}.\overrightarrow{n_b} = < \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} > \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{bmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} = \frac{1}{2}(\varepsilon_{xx} + \varepsilon_{yy}) + \varepsilon_{xy}$$

   Donc : $\boxed{\varepsilon_{xy} = \varepsilon_b - \frac{1}{2}(\varepsilon_{xx} + \varepsilon_{yy}) = 0.0013}$

$$\overset{=>}{\varepsilon}_{x,y,z} = \begin{bmatrix} 0.001 & 0.0013 & 0 \\ 0.0013 & 0.0008 & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix}$$

2) Comme on est dans un état plan de contrainte, on utilise les formules trouvées dans la première partie des questions de cours :

$$\sigma_{xx} = \frac{E}{1-\vartheta^2}\left(\varepsilon_{xx} + \vartheta\varepsilon_{yy}\right) = \frac{70000}{1-0.33^2}\left(0.001 + 0.33 * 0.0008\right) = \boxed{99.29}$$

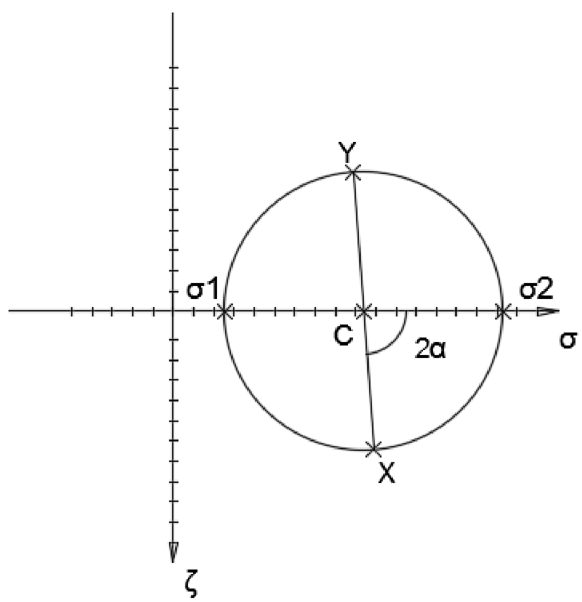
$$\sigma_{yy} = \frac{E}{1-\vartheta^2}\left(\varepsilon_{yy} + \vartheta\varepsilon_{xx}\right) = \frac{70000}{1-0.33^2}\left(0.0008 + 0.001 * 0.33\right) = \boxed{88.77}$$

$$\sigma_{xy} = \frac{E\varepsilon_{xy}}{1+\vartheta} = \frac{70000}{1+0.33}\,0.0013 = \boxed{68.42}$$

$$\overset{\Rightarrow}{\sigma} = \begin{bmatrix} 99.29 & 68.42 & 0 \\ 68.42 & 88.77 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3) On prend 2 points diamétralement opposés :

$$X \left| \begin{matrix} \sigma_{xx} = 99.29 \\ \sigma_{xy} = 68.42 \end{matrix} \right. \quad et \quad Y \left| \begin{matrix} \sigma_{yy} = 88.77 \\ -\sigma_{xy} = -68.42 \end{matrix} \right.$$



Le centre C a pour coordonnées : $\left(\frac{\sigma_{xx}+\sigma_{yy}}{2}, 0\right)$

Donc C (94.03 ; 0)

Le rayon R est tel que :

$$R = \sqrt{(\sigma_{xx} - C_x)^2 + \sigma_{xy}^2} = 68.62$$

Les contraintes principales :

$\sigma_1 = C_x - R = 94.03 - 68.62 = 25.41 MPa$

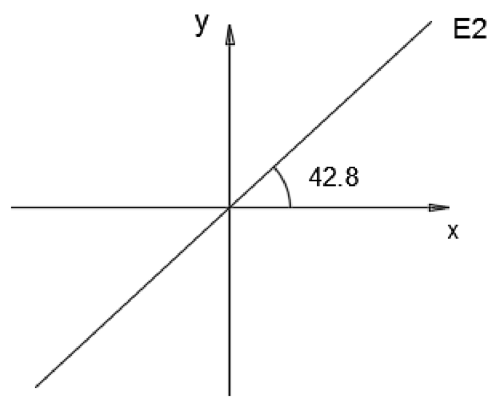
$\sigma_2 = C_x + R = 94.03 + 68.62 = 162.62\ MPa$

Directions principales :

$$sin2\alpha = \frac{\sigma_{xy}}{R} = \frac{68.42}{68.62} = 0.997$$

$2\alpha = \sin^{-1} 0.997 = 85.6$

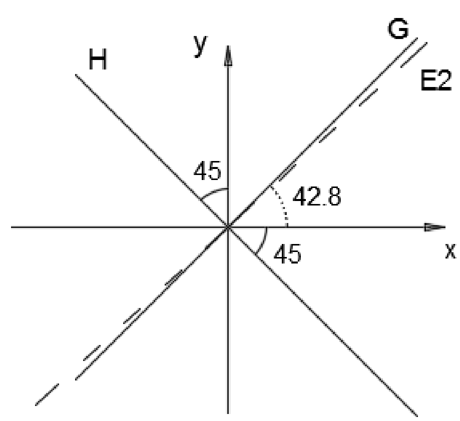
$\alpha = 42.8^{\circ}$



4) $\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{162.62 - 0}{2} = 81.31 MPa$

<u>Remarque :</u> Ne jamais oublier qu'il y a 3 contraintes dans un état plan de contrainte, dont celle perpendiculaire au plan est nulle.

Les directions du cisaillement maximal sont les bissectrices des directions principales



5) Les deux contraintes équivalentes :

$$\sigma_{Tresca} = \sigma_{max} - \sigma_{min} = 162.62\ MPa$$

$$\sigma_{Von\ Mise} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2} = 151.52MPa$$

Les deux coefficients de sécurité sont alors :

$$F_{S-Tresca} = \frac{\sigma_e}{\sigma_{Tresca}} = \frac{250}{162.62} = 1.537$$

$$F_{S-Von\ Mise} = \frac{\sigma_e}{\sigma_{Von\ Mise}} = \frac{250}{151.52} = 1.650$$

6) $U = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$ (sommation d'Einstein)

$$U = \frac{1}{2}\left(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + 2\sigma_{xy}\varepsilon_{xy} + 5*0\right)$$

$$U = \frac{1}{2}(99.29*0.001 + 88.77*0.0008 + 2*68.42*0.0013)$$

$$U = 0.174\ MPa$$

7) $$\begin{cases} \varepsilon_{zz} = \frac{\Delta l}{l} \\ \varepsilon_{zz} = \frac{-\vartheta}{E}\left(\sigma_{xx}+\sigma_{yy}\right) \end{cases}$$

Donc :

$$\Delta l = \frac{-\vartheta}{E}\left(\sigma_{xx}+\sigma_{yy}\right)l = \frac{-0.33}{70000}(99.29+88.77)*5 = -0.004433mm$$

Problème 2 :

$$\vec{\sigma}_{\overrightarrow{E1},\overrightarrow{E2},\overrightarrow{E3}} = \begin{bmatrix} 80 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{OU} \begin{bmatrix} 200 & 0 & 0 \\ 0 & 80 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{\sigma}_{x,y,z} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{OU} \begin{bmatrix} 120 & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

a) On utilise les invariants $I_{1\sigma}\ et\ I_{3\sigma}$:

$$I_{1\sigma} = 80 + 200 = 120 + \sigma_{yy}$$

$$\boxed{\sigma_{yy} = 280 - 120 = 160\ Mpa}$$

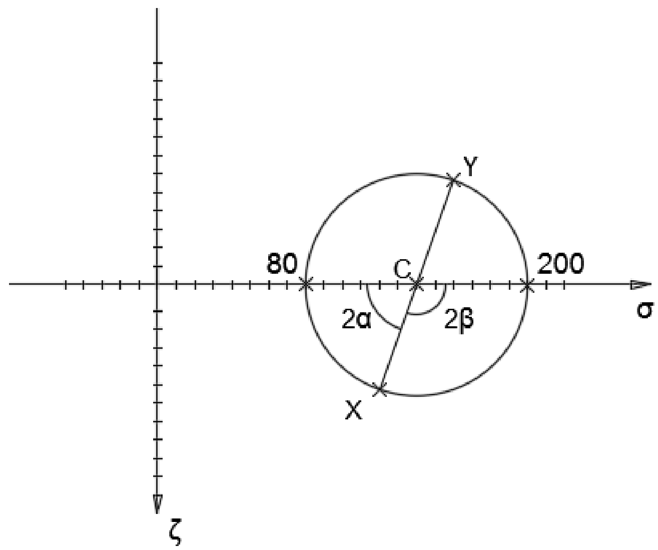
$I_{3\sigma} = 80 * 200 = 120 * 160 - {\sigma_{xy}}^2$

${\sigma_{xy}}^2 = -80 * 200 + 120 * 160 = 3200$

$\boxed{\sigma_{xy} = \pm\sqrt{3200} = \pm 40\sqrt{2}\ Mpa}$

b) Supposons $\sigma_{xy} > 0$

X= $\left| \begin{array}{l} \sigma_{xx} = 120 \\ \sigma_{xy} = 40\sqrt{2} \end{array} \right.$ Y= $\left| \begin{array}{l} \sigma_{yy} = 160 \\ -\sigma_{xy} = -40\sqrt{2} \end{array} \right.$



$$R = \frac{200 - 80}{2} = 60$$

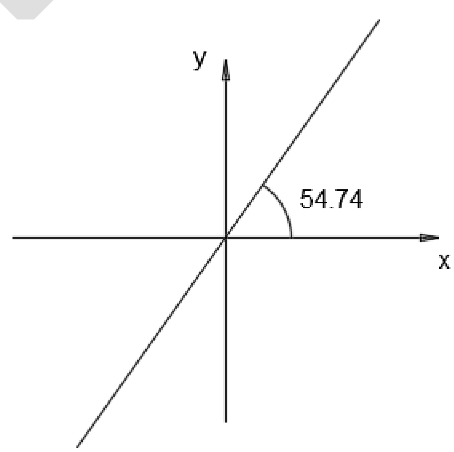
On cherche $2\beta = 180 - 2\alpha$

$\sin 2\alpha = \frac{\sigma_{xy}}{R} = \frac{40\sqrt{2}}{60} = \frac{2\sqrt{2}}{3}$

$2\alpha = \sin^{-1}(\frac{2\sqrt{2}}{3}) = 70{,}53$

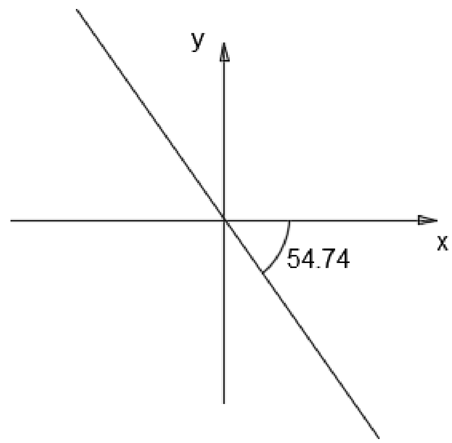
Donc $2\beta = 180 - 70.53 = 109{,}47$

$\beta = \frac{109{,}47}{2}$ =54,74

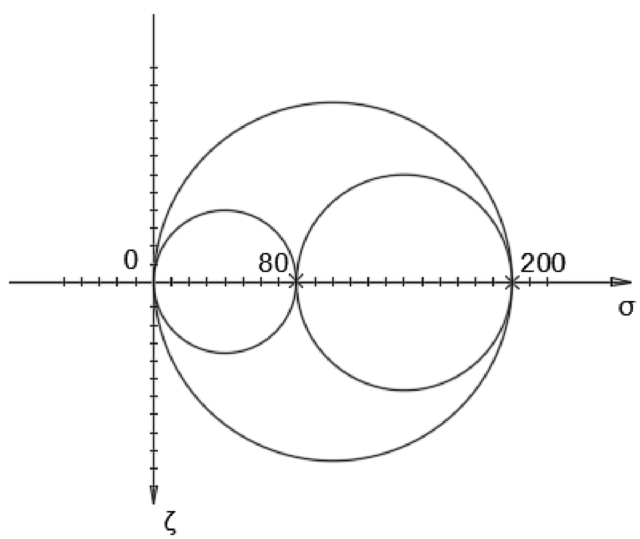


A noter bien que dans ce cas, on a supposé que $\sigma_{xy} > 0$.

Si $\sigma_{xy} > 0$, on aurait du remplacer X par Y dans le cercle de Mohr, d'où le résultat suivant :



c)



$\tau_{max} = \frac{\tau_{max} - \tau_{min}}{2} = \frac{200}{2} = 100Mpa$

$\sigma_{Von\ mise} = \frac{1}{\sqrt{2}}\sqrt{(200-80)^2 + 80^2 + 200^2} = 174{,}36Mpa$

Problème 3 :

$\sigma_{xx} = Ay$
$\sigma_{xy} = 2x$
$\sigma_{yy} = Bx + Cy$

Et on a le plan d'équation $x - 2y = 0$ de vecteur unitaire normal : $\vec{n} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$

D'une part, on n'a pas de force de volumes, les équations de l'équilibre se réduisent à :

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0$$

Soit :

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \quad \Rightarrow 0 + 0 = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \quad \Rightarrow 2 + C = 0 \ \Rightarrow C = -2$$

D'autre part, le plan $x - 2y = 0$ est libre de contraintes donc :

$$\overset{=>}{\sigma}\,\vec{n} = \vec{0}$$

$$\Rightarrow \begin{bmatrix} Ay & 2x & 0 \\ 2x & Bx - 2y & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{Ax}{2} & 2x & 0 \\ 2x & (B-1)x & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \frac{Ax}{2} - 4x = 0 \\ 2x - 2\left(B - \frac{1}{2}\right)x = 0 \end{cases}$$

$$\Rightarrow \begin{cases} Ax = 8x \\ x = (B-1)x \end{cases}$$

$$\Rightarrow \begin{cases} A = 8 \\ B = 2 \end{cases}$$

Pour $y = 2$:

$\sigma_{xx} = 8y = 12$
$\sigma_{xy} = 2x$
$\sigma_{yy} = 2x - 2y = 2x - 4$